Lecture 9

Sequence Tagging

Part 1: Sequence Tagging

Machine Learning for Structured Data
Vlad Niculae · LTL, UvA · `https://vene.ro/mlsd`

Outline:

Sequence Tagging

Given a sequence of n items $\boldsymbol{x} = (x_1, \ldots, x_n)$, assign to each of them one of K tags:

$$\boldsymbol{y} = (y_1, \ldots, y_n) \quad \text{where each} \quad y_i \in \{1, \ldots, K\}.$$

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Example 1: Part-of-speech (POS) tagging in NLP

	the	old	man	the	boat
$\boldsymbol{y}_a$	`det`	`adj`	`noun`	`det`	`noun`
$\boldsymbol{y}_b$	`det`	`noun`	`verb`	`det`	`noun`

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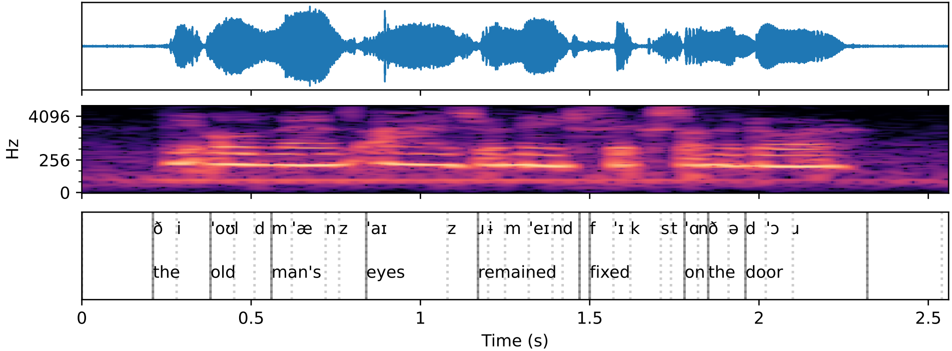
Example 2: Frame-level phoneme classification (may be part of speech recognition)



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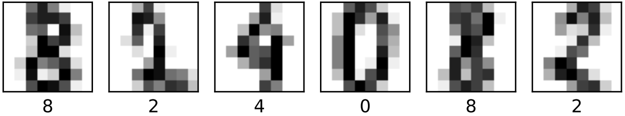
Example 3: Optical character recognition



Characterizing The Output Space

Given a sequence of n items $\boldsymbol{x} = (x_1, \ldots, x_n)$, assign to each of them one of K tags:

$$\boldsymbol{y} = (y_1, \ldots, y_n) \quad \text{where each} \quad y_i \in \{1, \ldots, K\}.$$

Input $\boldsymbol{x} = (x_1, \ldots, x_n)$, e.g., a sequence of words.

Output $\boldsymbol{y} = (y_1, \ldots, y_n)$, e.g., a sequence of part-of-speech tags.

For each data point (sentence), $|\boldsymbol{y}| = |\boldsymbol{x}|$; different data points have different lengths.

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For each data point (sentence), $|\boldsymbol{y}| = |\boldsymbol{x}|$; different data points have different lengths.

For fixed length n, some possible outputs:

- $(1, 1, \dots, 1, 1) \in \mathcal{Y}$
- $(1, 1, \dots, 1, 2) \in \mathcal{Y}$
- $(K, K, \dots, K, K) \in \mathcal{Y}$

How many in terms of n?

Part-Of-Speech Tags

	Tag	Description	Example
Open Class	**ADJ**	Adjective: noun modifiers describing properties	*red, young, awesome*
	ADV	Adverb: verb modifiers of time, place, manner	*very, slowly, home, yesterday*
	NOUN	words for persons, places, things, etc.	*algorithm, cat, mango, beauty*
	VERB	words for actions and processes	*draw, provide, go*
	PROPN	Proper noun: name of a person, organization, place, etc..	*Regina, IBM, Colorado*
	INTJ	Interjection: exclamation, greeting, yes/no response, etc.	*oh, um, yes, hello*
Closed Class Words	**ADP**	Adposition (Preposition/Postposition): marks a noun's spacial, temporal, or other relation	*in, on, by, under*
	AUX	Auxiliary: helping verb marking tense, aspect, mood, etc.,	*can, may, should, are*
	CCONJ	Coordinating Conjunction: joins two phrases/clauses	*and, or, but*
	DET	Determiner: marks noun phrase properties	*a, an, the, this*
	NUM	Numeral	*one, two, first, second*
	PART	Particle: a preposition-like form used together with a verb	*up, down, on, off, in, out, at, by*
	PRON	Pronoun: a shorthand for referring to an entity or event	*she, who, I, others*
	SCONJ	Subordinating Conjunction: joins a main clause with a subordinate clause such as a sentential complement	*that, which*
Other	**PUNCT**	Punctuation	; , ()
	SYM	Symbols like $ or emoji	$, %
	X	Other	asdf, qwfg

Figure 8.1 The 17 parts of speech in the Universal Dependencies tagset (Nivre et al., 2016a). Features can be added to make finer-grained distinctions (with properties like number, case, definiteness, and so on).

Source: Jurafsky and Martin, Speech and Language Processing. `https://web.stanford.edu/~jurafsky/slp3/8.pdf` ©Jurafsky and Martin

POS Tagging Evaluation

Evaluation: sequence-level accuracy

$$\frac{\sum_{i=1}^{N_{\text{valid}}} \boldsymbol{y}^{(i)} = \hat{\boldsymbol{y}}^{(i)}}{N_{\text{valid}}}$$

or micro-averaged tag accuracy (writing $n^{(i)} = |\boldsymbol{y}^{(i)}|$):

$$\frac{\sum_{i=1}^{N_{\text{valid}}} \sum_{j=1}^{n^{(i)}} y_j^{(i)} = \hat{y}_j^{(i)}}{\sum_{i=1}^{N_{\text{valid}}} n^{(i)}}$$

Example:

true:	PRO	VERB	NUM	NOUN	ADV
pred:	PRO	VERB	NUM	NOUN	PRO
words:	there	are	70	children	there

true:	INTJ
pred:	X
words:	eeeeek

Lecture 9

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Part 2: Different Scoring Models

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Outline:

Designing A Simple Scorer

Writing $\boldsymbol{y} = (y_1, \dots, y_n)$, take $\text{score}(\boldsymbol{y}) = \sum_j a_{j,y_j}$.

$\boldsymbol{A}$ is a matrix of scores, e.g., computed by a NN encoder.

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

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	the	old	man	the	boat
$\boldsymbol{y}_a$	det	adj	noun	det	noun
$\boldsymbol{y}_b$	det	noun	verb	det	noun

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	the	old	man	the	boat
$\boldsymbol{y}_a$	det	adj	noun	det	noun
$\boldsymbol{y}_b$	det	noun	verb	det	noun

score$(\boldsymbol{y}_a) =$

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	the	old	man	the	boat
$\boldsymbol{y}_a$	det	adj	noun	det	noun
$\boldsymbol{y}_b$	det	noun	verb	det	noun

$\text{score}(\boldsymbol{y}_a) = 21$

Designing A Simple Scorer

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score$(\boldsymbol{y}_a) = 21$

score$(\boldsymbol{y}_b) =$

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	the	old	man	the	boat
$\boldsymbol{y}_a$	det	adj	noun	det	noun
$\boldsymbol{y}_b$	det	noun	verb	det	noun

$\text{score}(\boldsymbol{y}_a) = 21$

$\text{score}(\boldsymbol{y}_b) = 17$

Designing A Simple Scorer

A first attempt:
separate classifier for each position.

1. embed and encode $\boldsymbol{x}$, eg, with a CNN.

$$(x_1, \ldots, x_n) \rightarrow (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)$$

2. For each position j, apply a classification head with K outputs. E.g.,

$$\boldsymbol{a}_j = \boldsymbol{W}^\top \boldsymbol{z}_j + \boldsymbol{b}$$

Think of $\boldsymbol{A}$ as a matrix with n rows and K columns, where $a_{j,c}$ is the score of assigning tag c at position j.

3. Writing $\boldsymbol{y} = (y_1, \ldots, y_n)$, take $\operatorname{score}(\boldsymbol{y}) = \sum_j a_{j,y_j}$.

```
words = [21, 79, 14]  # indices
emb = Embedding(vocab_sz, dim)
clf = Linear(dim, n_tags)

# optionally add RNN, CNN, whatever

Z = emb(words)  # (3 × dim)
A = clf(Z)      # (3 × n_tags)

# computing the score of a given tag sequence:
y = [2, 0, 2]

y_score = sum(A[i, yi]
              for y, yi in enumerate(y))

# or, if you want to be fancy/fast:
y_score = A[torch.arange(len(y)), y].sum()
```

Finding The Best sequence

With our $\text{score}(\boldsymbol{y}) = \sum_j a_{j,y_j}$, can we compute:

$$\max_{\boldsymbol{y} \in \mathcal{Y}} \text{score}(\boldsymbol{y})$$

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

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\begin{aligned}
&\max_{\boldsymbol{y} \in \mathcal{Y}} \text{score}(\boldsymbol{y}) \\
&= \max_{y_1 \in [K], \dots, y_n \in [K]} \text{score}\left([y_1, \dots, y_n]\right) \\
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So, $\arg\max_{\boldsymbol{y}} \text{score}(\boldsymbol{y})$ is made up of the tags selected independently at each position.

Normalizing Constant (log-sum-exp)

With our $\mathrm{score}(\boldsymbol{y}) = \sum_j a_{j,y_j}$, can we compute:

$$\log \sum_{\boldsymbol{y} \in \mathcal{Y}} \exp\left(\mathrm{score}(\boldsymbol{y})\right)$$

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Probabilistic interpretation: independence

$$
\begin{aligned}
\log \Pr(\boldsymbol{y}) &= \text{score}(\boldsymbol{y}) - \log \sum_{\boldsymbol{y}' \in \mathcal{Y}} \exp \text{score}(\boldsymbol{y}') \\
&= \sum_j \underbrace{\left(a_{j,y_j} - \log \sum_{k \in [K]} \exp a_{j,k} \right)}_{\log \Pr(y_j)}
\end{aligned}
$$

Fully-Local vs. Fully-Global

For sequence tagging, the separable (fully-local) score

$$\text{score}(\boldsymbol{y}) = \sum_j a_{j,y_j}$$

amounts to applying a probabilistic classifier to each of the n positions separately!
(any “magic” comes from the feature represtntation / neural net encoder.)

Can we design a richer $\text{score}(\boldsymbol{y})$ taking into account the sequential structure of $\boldsymbol{y}$?

Fully-Local vs. Fully-Global

Entirely global model: like classification, where *each possible sequence* is a class.

$\boldsymbol{y}$	$\text{score}(\boldsymbol{y})$
`det det det det det`	-1000
`det det det det noun`	-940
`det det det det verb`	-800
$\cdots$	
`det noun verb det noun`	400
$\cdots$	
`verb verb verb verb verb`	-1100

As expressive as possible: score is any function of the sequence.

Fully-Local vs. Fully-Global

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But completely intractable: $O(K^n)$ time and space.

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...	
det noun verb det noun	400
...	
verb verb verb verb verb	-1100

As expressive as possible: score is any function of the sequence.

But completely intractable: $O(K^n)$ time and space.

Structure output prediction is about the space in between these two extremes.

Idea: scoring transitions between adjacent tags

$$\mathrm{score}(\boldsymbol{y}) = \sum_{j=1}^{n} a_{j,y_j} + \sum_{j=2}^{n} t_{y_{j-1},y_j}$$

For example, $\mathrm{score}(\texttt{[NOUN, DET, VERB]}) = +a_{2,\mathrm{DET}}\, a_{1,\mathrm{NOUN}} + a_{3,\mathrm{VERB}} + t_{\mathrm{NOUN},\mathrm{DET}} + t_{\mathrm{DET},\mathrm{VERB}}$

Scoring Transitions Between Tags

A rich scorer that takes into account the sequential nature of $\boldsymbol{y}$ while still allowing efficient computation:

scoring transitions between adjacent tags

$$\operatorname{score}(\boldsymbol{y}) = \sum_{j=1}^{n} a_{j,y_j} + \sum_{j=2}^{n} t_{y_{j-1},y_j}$$

For example, $\operatorname{score}(\texttt{[NOUN, DET, VERB]}) = a_{1,\mathrm{NOUN}} + a_{2,\mathrm{DET}} + a_{3,\mathrm{VERB}} + t_{\mathrm{NOUN},\mathrm{DET}} + t_{\mathrm{DET},\mathrm{VERB}}$

Sequence Modeling With Transition Scores

$$\text{score}(\boldsymbol{y}) = \sum_{j=1}^{n} a_{j,y_j} + \sum_{j=2}^{n} t_{y_{j-1},y_j}$$

The tag scores $\boldsymbol{A} \in \mathbb{R}^{n \times K}$ can be computed as before (e.g., with a convnet.)

The transition scores $\boldsymbol{T} \in \mathbb{R}^{K \times K}$:

- could be a learned parameter. (size does not depend on n)
- could be predicted by the neural net as a function of $\boldsymbol{x}$.

Unlike in the separable case, with transition scores, we no longer get n parallel classifiers: the different tags impact one another. (This makes the model more expressive and more interesting.)

Lecture 9

Sequence Tagging

Part 3: Sequence Tagging Algorithms

Machine Learning for Structured Data

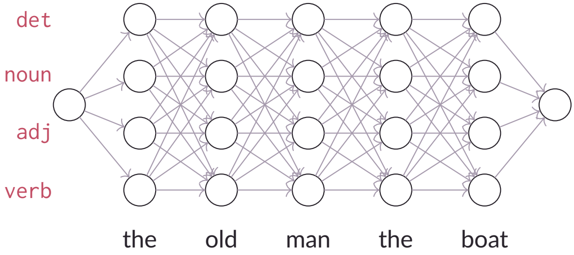
Vlad Niculae · LTL, UvA · https://vene.ro/mlsd

Outline:

1. Sequence Tagging
   - Definition and examples
   - Evaluation
2. Different Scoring Models
   - A Simple Scoring Function
   - A Better Scoring Model
3. **Sequence Tagging Algorithms**
   - Dynamic Programming For Sequence Tagging
   - Putting It All Together

Sequence Tagging As A DAG

$$\text{score}(\boldsymbol{y}) = \sum_{j=1}^{n} a_{j,y_j} + \sum_{j=2}^{n} t_{y_{j-1},y_j}$$



$G = (V, E, w)$ where:

$$V = \{(j,c) : j \in [n], c \in [K]\} \cup \{s, t\}$$

$$E = \{(j-1, c') \to (j,c) : j \in [2,n], c, c' \in [K]\} \cup \{s \to (1,c) : c \in [K]\} \cup \{(n,c) \to t : c \in [K]\}$$

$$w\left((j-1, c') \to (j,c)\right) = a_{j,c} + t_{c',c}$$

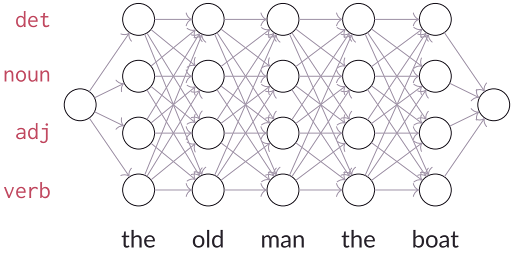
$$w(s \to (1,c)) = a_{1,c}$$

$$w((n,c) \to t) = 0$$

$|V| \in \Theta(nK)$; $\quad |E| \in \Theta(nK^2)$

Topological ordering?

Viterbi For Sequence Tagging



General Viterbi (reminder sketch)

initialize $m_1 \leftarrow 0$
for $i = 2, \ldots, n$ **do**

$$m_i \leftarrow \max_{j \in P_i} \left(m_j + w(ji)\right)$$

$$\pi_i \leftarrow \arg\max_{j \in P_i} \left(m_j + w(ji)\right)$$

follow backpointers to get best path

Viterbi for sequence tagging

input: Unary scores $\boldsymbol{A}$ ($n \times K$ array)
Transition scores $\boldsymbol{T}$ ($K \times K$ array)

Forward: compute scores recursively

$m_{1c} = a_{1c}$ for all $c \in [K]$
for $j = 2$ to n **do**
for $c = 1$ to K **do**

$$m_{j,c} \leftarrow \max_{c' \in [K]} \left(m_{j-1,c'} + a_{j,c} + t_{c',c}\right)$$

$$\pi_{j,c} \leftarrow \arg\max_{c' \in [K]} \left(m_{j-1,c'} + a_{j,c} + t_{c',c}\right)$$

$f^\star = \max_{c' \in [K]} m_{n,c'}$

Backward: follow backpointers

$y_n = \arg\max_{c'} m_n(c')$
for $j = n - 1$ **down to** 1 **do**
$y_j = \pi_{j+1, y_{j+1}}$

output: $f^\star$ and $\boldsymbol{y}^\star = [y_1, \ldots, y_n]$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & & & & \\ \text{old} & & & & \\ \text{man} & & & & \\ \text{the} & & & & \\ \text{boat} & & & & \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & & & & \\ \text{man} & & & & \\ \text{the} & & & & \\ \text{boat} & & & & \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & & & & \\ \text{man} & & & & \\ \text{the} & & & & \\ \text{boat} & & & & \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 1 & & & \\ \text{man} & & & & \\ \text{the} & & & & \\ \text{boat} & & & & \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 1 & & & \\ \text{man} & & & & \\ \text{the} & & & & \\ \text{boat} & & & & \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$\boldsymbol{M} =$

	det	noun	adj	verb
the	5	0	0	0
old	1	9		
man				
the				
boat				

unary and transition scores:

$\boldsymbol{A} =$

	det	noun	adj	verb
the	5	0	0	0
old	0	1	3	0
man	0	3	0	1
the	5	0	0	0
boat	0	5	0	0

$\boldsymbol{T} =$

	det	noun	adj	verb
det	−4	3	2	−1
noun	−3	−2	−1	2
adj	−2	2	1	1
verb	1	−1	0	0

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 1 & 9 & 10 & 4 \\ \text{man} & & & & \\ \text{the} & & & & \\ \text{boat} & & & & \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

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Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 1 & 9 & 10 & 4 \\ \text{man} & 8 & 15 & 11 & 12 \\ \text{the} & 18 & 13 & 14 & 17 \\ \text{boat} & 18 & 26 & 20 & 17 \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{c|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 1 & 9 & 10 & 4 \\ \text{man} & 8 & 15 & 11 & 12 \\ \text{the} & 18 & 13 & 14 & 17 \\ \text{boat} & 18 & 26 & 20 & 17 \end{array}$$

To find the best tag sequence $\boldsymbol{y}^\star$, keep track of the path.

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

Viterbi For Sequence Tagging: Example

$m_{j,c}$ is stored as a matrix $\boldsymbol{M}$, same shape as $\boldsymbol{A}$.

Apply $m_{1,c} = a_{1,c}$ to get the first row: (copied from $\boldsymbol{A}$)

Then iteratively: $m_{j,c} = \max_{c' \in [K]} m_{j-1,c'} + a_{j,c} + t_{c',c}$

At the end, take the maximum over the last row.

$$\boldsymbol{M} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & \boxed{5} & 0 & 0 & 0 \\ \text{old} & 1 & \boxed{9} & 10 & 4 \\ \text{man} & 8 & 15 & 11 & \boxed{12} \\ \text{the} & \boxed{18} & 13 & 14 & 17 \\ \text{boat} & 18 & \boxed{26} & 20 & 17 \end{array}$$

To find the best tag sequence $\boldsymbol{y}^\star$, keep track of the path.

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{r|cccc} & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\ \hline \texttt{det} & -4 & 3 & 2 & -1 \\ \texttt{noun} & -3 & -2 & -1 & 2 \\ \texttt{adj} & -2 & 2 & 1 & 1 \\ \texttt{verb} & 1 & -1 & 0 & 0 \end{array}$$

The Two Main Recurrences Of Sequence Tagging:

(Dynamic programming applied to the sequence tagging DAG)

$$
\begin{aligned}
m_{j,c} &= \max_{c' \in [K]} \left(m_{j-1,c'} + a_{jc} + t_{c'c} \right), \\
q_{j,c} &= \log \sum_{c' \in [K]} \exp \left(q_{j-1,c'} + a_{jc} + t_{c'c} \right).
\end{aligned}
$$

The Forward Algorithm

Forward algorithm for sequence tagging

input: Unary scores $\boldsymbol{A}$ ($n \times K$ array)
Transition scores $\boldsymbol{T}$ ($K \times K$ array)

Forward: compute scores recursively
$q_{1,c} = a_{1,c}$ for all $c \in [K]$
for $j = 2$ **to** n **do**
  for $c = 1$ **to** K **do**
    $q_{j,c} = \log \sum_{c' \in [K]} \exp\left(q_{j-1,c'} + a_{j,c} + t_{c',c}\right)$
return $\log Z = \log \sum_{c' \in [K]} \exp\left(q_{n,c'}\right)$

	the	old	man	the	boat	
$\boldsymbol{y}_a$	det	adj	noun	det	noun	$\text{score}(\boldsymbol{y}_a) = 25$
$\boldsymbol{y}_b$	det	noun	verb	det	noun	$\text{score}(\boldsymbol{y}_b) = 26$
$\boldsymbol{y}_c$	noun	noun	noun	noun	noun	$\text{score}(\boldsymbol{y}_c) = 1$

Applying the Forward algorithm yields

$$\boldsymbol{Q} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5.00 & 0.00 & 0.00 & 0.00 \\ \text{old} & 1.73 & 9.00 & 10.00 & 4.19 \\ \text{man} & 8.18 & 15.01 & 11.05 & 12.70 \\ \text{the} & 18.88 & 13.92 & 14.37 & 17.03 \\ \text{boat} & 18.08 & 26.88 & 20.90 & 18.38 \end{array}$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{det} & -4 & 3 & 2 & -1 \\ \text{noun} & -3 & -2 & -1 & 2 \\ \text{adj} & -2 & 2 & 1 & 1 \\ \text{verb} & 1 & -1 & 0 & 0 \end{array}$$

	the	old	man	the	boat	
$\boldsymbol{y}_a$	det	adj	noun	det	noun	$\text{score}(\boldsymbol{y}_a) = 25$
$\boldsymbol{y}_b$	det	noun	verb	det	noun	$\text{score}(\boldsymbol{y}_b) = 26$
$\boldsymbol{y}_c$	noun	noun	noun	noun	noun	$\text{score}(\boldsymbol{y}_c) = 1$

Applying the Forward algorithm yields

$$\boldsymbol{Q} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5.00 & 0.00 & 0.00 & 0.00 \\ \text{old} & 1.73 & 9.00 & 10.00 & 4.19 \\ \text{man} & 8.18 & 15.01 & 11.05 & 12.70 \\ \text{the} & 18.88 & 13.92 & 14.37 & 17.03 \\ \text{boat} & 18.08 & 26.88 & 20.90 & 18.38 \end{array}$$

$$\log Z \approx 26.885$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{det} & -4 & 3 & 2 & -1 \\ \text{noun} & -3 & -2 & -1 & 2 \\ \text{adj} & -2 & 2 & 1 & 1 \\ \text{verb} & 1 & -1 & 0 & 0 \end{array}$$

	the	old	man	the	boat	
$\boldsymbol{y}_a$	det	adj	noun	det	noun	$\text{score}(\boldsymbol{y}_a) = 25$
$\boldsymbol{y}_b$	det	noun	verb	det	noun	$\text{score}(\boldsymbol{y}_b) = 26$
$\boldsymbol{y}_c$	noun	noun	noun	noun	noun	$\text{score}(\boldsymbol{y}_c) = 1$

Applying the Forward algorithm yields

$$\boldsymbol{Q} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5.00 & 0.00 & 0.00 & 0.00 \\ \text{old} & 1.73 & 9.00 & 10.00 & 4.19 \\ \text{man} & 8.18 & 15.01 & 11.05 & 12.70 \\ \text{the} & 18.88 & 13.92 & 14.37 & 17.03 \\ \text{boat} & 18.08 & 26.88 & 20.90 & 18.38 \end{array}$$

$$\log Z \approx 26.885$$

$$\log P(\boldsymbol{y}_a) = \text{score}(\boldsymbol{y}_a) - \log Z = 25 - 26.885 = -1.885$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{det} & -4 & 3 & 2 & -1 \\ \text{noun} & -3 & -2 & -1 & 2 \\ \text{adj} & -2 & 2 & 1 & 1 \\ \text{verb} & 1 & -1 & 0 & 0 \end{array}$$

	the	old	man	the	boat	
$\boldsymbol{y}_a$	det	adj	noun	det	noun	$\text{score}(\boldsymbol{y}_a) = 25$
$\boldsymbol{y}_b$	det	noun	verb	det	noun	$\text{score}(\boldsymbol{y}_b) = 26$
$\boldsymbol{y}_c$	noun	noun	noun	noun	noun	$\text{score}(\boldsymbol{y}_c) = 1$

Applying the Forward algorithm yields

$$\boldsymbol{Q} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5.00 & 0.00 & 0.00 & 0.00 \\ \text{old} & 1.73 & 9.00 & 10.00 & 4.19 \\ \text{man} & 8.18 & 15.01 & 11.05 & 12.70 \\ \text{the} & 18.88 & 13.92 & 14.37 & 17.03 \\ \text{boat} & 18.08 & 26.88 & 20.90 & 18.38 \end{array}$$

$$\log Z \approx 26.885$$

$$\log P(\boldsymbol{y}_a) = \text{score}(\boldsymbol{y}_a) - \log Z = 25 - 26.885 = -1.885$$

$$\log P(\boldsymbol{y}_b) = \text{score}(\boldsymbol{y}_b) - \log Z = 26 - 26.885 = -0.885$$

unary and transition scores:

$$\boldsymbol{A} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{the} & 5 & 0 & 0 & 0 \\ \text{old} & 0 & 1 & 3 & 0 \\ \text{man} & 0 & 3 & 0 & 1 \\ \text{the} & 5 & 0 & 0 & 0 \\ \text{boat} & 0 & 5 & 0 & 0 \end{array}$$

$$\boldsymbol{T} = \begin{array}{c|cccc} & \text{det} & \text{noun} & \text{adj} & \text{verb} \\ \hline \text{det} & -4 & 3 & 2 & -1 \\ \text{noun} & -3 & -2 & -1 & 2 \\ \text{adj} & -2 & 2 & 1 & 1 \\ \text{verb} & 1 & -1 & 0 & 0 \end{array}$$

	the	old	man	the	boat	
$\boldsymbol{y}_a$	det	adj	noun	det	noun	$\text{score}(\boldsymbol{y}_a) = 25$
$\boldsymbol{y}_b$	det	noun	verb	det	noun	$\text{score}(\boldsymbol{y}_b) = 26$
$\boldsymbol{y}_c$	noun	noun	noun	noun	noun	$\text{score}(\boldsymbol{y}_c) = 1$

Applying the Forward algorithm yields

$$
\boldsymbol{Q} = \begin{array}{r|cccc}
 & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\
\hline
\text{the} & 5.00 & 0.00 & 0.00 & 0.00 \\
\text{old} & 1.73 & 9.00 & 10.00 & 4.19 \\
\text{man} & 8.18 & 15.01 & 11.05 & 12.70 \\
\text{the} & 18.88 & 13.92 & 14.37 & 17.03 \\
\text{boat} & 18.08 & 26.88 & 20.90 & 18.38
\end{array}
$$

$$
\begin{aligned}
\log Z &\approx 26.885 \\
\log P(\boldsymbol{y}_a) = \text{score}(\boldsymbol{y}_a) - \log Z &= 25 - 26.885 = -1.885 \\
\log P(\boldsymbol{y}_b) = \text{score}(\boldsymbol{y}_b) - \log Z &= 26 - 26.885 = -0.885 \\
\log P(\boldsymbol{y}_c) = \text{score}(\boldsymbol{y}_c) - \log Z &= 1 - 26.885 = -25.885
\end{aligned}
$$

unary and transition scores:

$$
\boldsymbol{A} = \begin{array}{r|cccc}
 & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\
\hline
\text{the} & 5 & 0 & 0 & 0 \\
\text{old} & 0 & 1 & 3 & 0 \\
\text{man} & 0 & 3 & 0 & 1 \\
\text{the} & 5 & 0 & 0 & 0 \\
\text{boat} & 0 & 5 & 0 & 0
\end{array}
$$

$$
\boldsymbol{T} = \begin{array}{r|cccc}
 & \texttt{det} & \texttt{noun} & \texttt{adj} & \texttt{verb} \\
\hline
\texttt{det} & -4 & 3 & 2 & -1 \\
\texttt{noun} & -3 & -2 & -1 & 2 \\
\texttt{adj} & -2 & 2 & 1 & 1 \\
\texttt{verb} & 1 & -1 & 0 & 0
\end{array}
$$

Putting It All Together

At this point, we have all the ingredients needed to train a probabilistic sequence tagger with transition scores!

1. Receiving an input sequence $\boldsymbol{x}$, the model returns unary and transition scores $\boldsymbol{A}$ and $\boldsymbol{T}$.
2. If we're at test time:
   run Viterbi to get predicted sequence; compute accuracies etc.
3. If training time:
   run Forward algorithm to compute the training objective
   $-\log P(\boldsymbol{y} \mid \boldsymbol{x}) = -\operatorname{score}(\boldsymbol{y}) + \log \sum_{\boldsymbol{y}' \in \mathcal{Y}} \exp \operatorname{score}(\boldsymbol{y}')$.

This probabilistic model is often known as a Linear-Chain Conditional Random Field.

(Historically, Linear-Chain CRFs didn't use neural net scorers, but the math doesn't change. Today I prefer to teach it this way.)